Indian Statistical Institute, Bangalore

B. Math.(hons.), Third Year, First Semester

Probability-III

End term Back-paper Examination

Maximum marks: 50 Time: 3 hours

Answer any 5, each question carries 10 marks.

Kindly ensure your writing is clear and if you are using any results please provide as many details as you can.

1. Let Q be a non-empty set. Let $\mathcal{F}_0$ be the collection of all subsets $A \subseteq Q$ such that either A or A^c is finite.

   (a) Show that $\mathcal{F}_0$ is a field. Further, define for $E \in \mathcal{F}_0$ the set function P by

   $$P(E) = \begin{cases} 0, & \text{if } E \text{ is finite,} \\ 1, & \text{if } E^c \text{ is finite.} \end{cases}$$

   (b) If Q is countably infinite, show that P is finitely additive but not σ-additive.

   (c) If Q is uncountable, show that P is σ-additive on $\mathcal{F}_0$.

2. (a) Let $X = (X_1, \ldots, X_n)$. Show that X is an $\mathbb{R}^n$-valued random variable if and only if $X_1, \ldots, X_n$ are (real-valued) random variables. How does $\sigma(X)$ relate to $\sigma(X_1), \ldots, \sigma(X_n)$?

   (b) Let $(Q, \mathcal{B}, P)$ be $([0,1], \mathcal{B}([0,1]), \lambda)$, where λ is the Lebesgue measure on $[0,1]$. Define the process $\{X_t, 0 \leq t \leq 1\}$ by

   $$X_t(\omega) = \begin{cases} 1, & \text{if } t \neq \omega, \\ 0, & \text{if } t = \omega. \end{cases}$$

   Show that each X_t is a random variable. What is the σ-field generated by $\{X_t, 0 \leq t \leq 1\}$?

3. (a) Suppose $T : (\Omega_1, \mathcal{B}_1) \to (\Omega_2, \mathcal{B}_2)$ is a measurable mapping and X is a random variable on Ω_1. Show that $X \in \sigma(T)$ if and only if there exists a random variable Y on $(\Omega_2, \mathcal{B}_2)$ such that

   $$X(\omega_1) = Y(T(\omega_1)), \quad \forall \omega_1 \in \Omega_1.$$

(b) Suppose $\{X_n : n \geq 1\}$ are independent random variables. Show that

$$P(\sup X_n < \infty) = 1$$

if and only if

$$\sum_{n=1}^{\infty} P(X_n > M) < \infty, \quad \text{for some } M.$$

4. (a) Suppose $\{X_n : n \geq 1\}$ are i.i.d. random variables with

$$P(X_n = 0) = P(X_n = 2) = \frac{1}{2}.$$

Show that $\sum_{n=1}^{\infty} \frac{X_n}{3^n}$ converges almost surely.

(b) Suppose that $f \in C([0,1];\mathbb{C})$, and for $x \in [0,1]$, define

$$p_n(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Then, show that

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f(x) - p_n(x)| = 0.$$

5. (a) Suppose X and Y are i.i.d. with mean zero and variance 1, and suppose further that $(X+Y)/\sqrt{2} \stackrel{d}{=} X \stackrel{d}{=} Y$. Then, show that both X and Y have $N(0,1)$ distribution.

(b) Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $X_n \sim \text{Binomial}(N_n, p_n)$ for all $n \geq 1$. Suppose that as $n \to \infty$, $N_n \to \infty$, $p_n \to 0$, and $N_n p_n \to \lambda$, where $\lambda \in (0,\infty)$. Then $X_n \xrightarrow{d} X \quad$ where $X \sim \text{Poisson}(\lambda)$.

6. If $\mu_n \in Prob(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $\mu_n \to \mu$ vaguely for some Borel measure, then $\mu(\mathbb{R}^d) = 1$ iff $\{\mu_n\}_{n=1}^{\infty}$ is tight.

7. A fair 6-sided die with faces numbered from 1 to 6 is rolled repeatedly, and for $n \in \mathbb{N}$, the outcome of the n-th roll is denoted by Y_n (it is assumed that $\{Y_n\}_{n\in\mathbb{N}}$ are independent of each other). For $n \in \mathbb{N}_0$, let X_n be the remainder (taken in the set $\{0,1,2,3,4\}$) left after the sum $\sum_{k=1}^{n} Y_k$ is divided by 5, i.e., $X_0 = 0$, and $X_n = \left(\sum_{k=1}^{n} Y_k\right) \mod 5, \quad$ for $n \in \mathbb{N}$, making $\{X_n\}_{n\in\mathbb{N}_0}$ a Markov chain on the state space $\{0,1,2,3,4\}$. Then,

(a) Write down the transition matrix of the chain.

(b) Classify the states, separating recurrent from transient ones, compute the period of each state. What will be the transition matrix in canonical form?